COMPENSATORY LOGIC: A FUZZY APROACH TO DECISION MAKING

ABSTRACT

The flexibility and adaptability of Fuzzy Logic are convenient qualities to Decision Making. Its capacity to elaborate linguistic models could be very useful to solve real problems getting a better communication with Decision Makers and Experts. Many revealing studies has been accomplished on Multi-valued Logics, including the explorations of a variety of operators; however, the real possibility to incorporate expert knowledge, and Decision Maker subjectivity in practical models still being limited. This lack is crucial; the necessity of better axiomatic approaches and practical capacity is evident. The aim of this paper is to provide a new axiomatic development on Multi-valued Logic, with applications in Decision Making. The proposal disregards the classical point of view of norm and conorm,. Existential and Universal quantifiers are defined consequently and Propositional Bivalent Classic Calculus is introduced from this Logical System.

Keywords: Fuzzy Logic, Management, Decision Making, Multivalued Logic.

1. INTRODUCTION

To combine the knowledge comprised in the literature with that contained in expert's brains may be acknowledged as a clear demand for rational decision-making in complex and dynamic environments (French 1986, Ostanello 1984). The conditions that modern managerial practice experiments, such as breakthroughs in technology of information, and the prescriptions of the new managerial paradigms demand the existence of more flexible mathematical models. Thus encompassing the subtleties of knowledge. This may enhance the organizational intelligence in a way that its strategic goals become more attainable as a result of improvements in tactical and operative decision-making processes.

The application of the Fuzzy Logic approach has expanded very rapidly to obtain models in non formalized sciences; and remarkable progress has been made in the developing of computing systems for several purposes, including finances and enterprise direction (Von Altrock 1995, Kauffman/Gil Aluja, 1990, Gil Aluja 1996). Its flexibility permits the effective interpretation of natural language, as expressed in any situation, to construct formal models, and to render conclusions on the basis of these models.

Even though the main aspects of this theory are remarkable, several pragmatic concerns show the necessity to perfect. One of the most well-known applications of Fuzzy logic is Automatic Control (Passino 1998, Reznik 1997). It could be said, that the use of simple rules, instead of intricate procedural patterns has shown better results in this practice. The usage of average methods of defuzzification may be recognized as evidencing the appropriateness and relevance of the new logic-system development which essence is compensation.

It can be argued that in the current practice of managerial decision-making, the use of complex predicates, such as those coming from exchanges with experts, shows a tendency to construct complex and subtly frameworks. This puts demands on the possibilities of a logical approach in order to cope with the structural complexity of businesses and their unstable environments.

On the other hand, the assignment of truth-values to predicates through application of diverse multi-value logics, lack some desirable properties. One of these concerns their sensibility to changes in truth-values of the basic predicates, or the 'verbal meaning' of the truth-values of one agent. Likewise, the variations associated with the selection of the connective implication are significant, and the theory proposes a high number of diverse operators that are not strictly determined by specific conjunctions and disjunctions. The aim of this paper is to propose a logical system that can cope with these drawbacks.

2. BASIC NOTIONS OF FUZZY LOGIC

In Boolean logic a predicate p is a mapping from the universe set X to {0, 1}. For example, the sentence 'x is a friend of y' admits a model in which, according to this logic, the predicate p from the set of pairs (x, y) to {0,1}, assigns 1 if x is effectively a friend of y and 0 if it cannot be assured that x is a friend of y.

The propositional connectives ∧ , ∨ , y, ¬, symbolize operations on sentences.

- p ∧ q is true when and only when both p and q are true. It is called conjunction, and symbolizes the inclusive use of "or" in natural language.
- p ∨ q is false when and only when both p and q are false. It is called disjunction, and symbolizes the use of "and" in natural language.
- ¬p is true when p is false, and conversely. It is called the negation of p.

As they are defined in regard to truth values, their functional nature can be addressed as mappings from {0, 1}×{0, 1} and ({0;1} for ¬p) to {0;1} (Aranda,1993)

Thus, for instance, if p(x, y) symbolizes the sentence 'x is a friend of y', then the sentence 'x is a friend of y, but he is not a friend of x' should be symbolized by the predicate p(x, y) ∧ ¬p(y, x).

The set of predicates built up by the application of ∨, ∧, and ¬ satisfy a number of properties which render the structure of the classical Boolean Algebra. Each statement composed by propositional connectives determines a truth function, which assigns its truth-values by virtue of the truth-values of the component statements, and the truth tables of the propositional connectives. One of the basic notions of propositional algebra is the 'Law of the Excluded Middle', which asserts the tautological character of the statement ¬(p∨¬p).

In contrast to the Law of the Excluded Middle, the principle of gradual simultaneity is ascertained within new logical approaches for which a predicate is a mapping from the universe X to the interval [0, 1]; instead of the classical set {0, 1}. This definition satisfies the Boolean 'Propositional Calculus', excepting the law of the exclusion, and its implications.

The exertion of these logics in practical problems requires the construction of category scales, or classifications, which can aid experts to assign truth values; or specify the dependence of predicates on proxy attributes to permit their direct assignment. A widely used scale is the following:
0: false; 0.1: nearly false; 0.2: very false; 0.3: somewhat false; 0.4: more false than true; 0.5: as true as false; 0.6: more true than false; 0.7: somewhat true; 0.8: very true; 0.9: nearly true; 1: true.

One multi-valued logic arises from the following definitions:

- u(p∧q) = u(p).u(q)
- u(p∨q) = u(p) + u(q) – u(p).u(q)
- u(¬p) =1– u(p)
-

where u(p) is the truth value of p.

The logic so defined does not satisfy transitivity, but it satisfies commutativity, associativity, and De Morgan's Laws. This type of logic is usually referred to as probabilistic logic. The definition of connectives is accomplished by the probabilistic expressions for union and intersection of events. However, the assignment does not assume any interpretation in terms of probability.

In addition, connectives ∧ y ∨ satisfy the following properties.

- Any increment in the truth values of p and q cause increments in the truth values of p∧q, whenever one of the truth values remains at level 0.
- Any increment in the truth values of p, or of q, cause increments in the truth values of p∨ q, whenever any of the truth values remains at level 1.

The latter reveals an 'attractive' sensibility, which, in regard to the normative approach to decision analysis, can be related to the axiom of continuity. Yet this property allows the possibility to modelling veto conditions, as considered by the so-called European approach. This feature makes it rather convenient for structuring and solving problems concerning ranking construction or alternative selection. The failure to satisfy the idempotent property causes the loss of verbal significance, or its attainment to a category scale. This feature disregards the approach to be used in classificatory or evaluation problems.

Considering:

- u(p ∧q) = min(u(p), u(q))
- u(p∨q) = max(u(p), u(q))
- u(¬p) =1– u(p)

the most used logic is obtained. It is, of course, the Fuzzy Logic system. Certainly, this is an extension of the two-value logics; moreover it is the only associative and idempotent system that satisfies properties (1) and (2):

$$v(p \wedge q) \leq \min(v(p), v(q)) \qquad (1)$$

$$v(p \vee q) \geq \max(v(p), v(q)) \qquad (2)$$

These statements are sufficient to assure the cardinality of the truth values; which is convenient for classification and appraising problems. However, the values of the compound predicates may remain unaffected by important modifications exerted on values of the basic predicates; thus making the approach unsuitable for ranking or selection problems.

The operators * that satisfy
are called average operators.

$$\min(v(p), v(q)) \leq v(p * q) \leq \max(v(p), v(q)) \qquad (3)$$

The definition of operators for implication is rather diverse (Dubois D. /Prade H. 1980); for example, the definition $p \to q = \neg p \vee q$, and the so-called Zadeh implication $p \to q = \neg p \vee (p \wedge q)$, which is widely utilized in control theory. Similar to Fuzzy Logic (in a narrow sense), the latter satisfies the classical inductive implication 'modus ponens' in terms of Zadeh's definition of a well-formed deductive structure (Dubois D. /Prade H. 1980, Zadeh 1965). Thus, the equivalence is defined by $p \leftrightarrow q = (p \to q) \wedge (q \to p)$ and the respective universal and existential quantifiers over X are defined by:

$$\exists x\, p(x) \;=\; \underset{x \in X}{\vee} (p(x))$$

$$\forall x\, p(x) = \underset{x \in X}{\wedge} (p(x))$$

These definitions convey the virtues and defects of the used conjunction and disjunction connectives.

3. COMPENSATORY LOGIC

In this theory, conjunction is usually defined as a continuous, associative and symmetric connective that satisfies (1); and the disjunction is defined as an operator implied by De Morgan's laws, according to the definition of conjunction. In such circumstances, conjunction satisfies the t-norm property, and disjunction satisfies the t-conorm property.

It should be noticed that properties 1 and 2 lead to the conclusion that the truth-value of the conjunction is equal or less than those of its components; and the truth value of the disjunction is equal or greater than those of its components. The rejection of these properties constitutes the basic idea of the Compensatory Logic. In contrast, it bears the notion that an increase or decrease of the truth value of the conjunction or disjunction, as a result of changes in the truth value of one of its components, can be compensated by an increase or decrease, respectively, in another component. This notion yields a very sensible Multi-value Logic, which also maintains the category value of the truth values. This makes it especially useful for selection problems, but it is also convenient for ranking, appraising, and classificatory purposes.

In this logic, conjunction should be a continuous operator from $[0,1]^n$ to $[0,1]$, such that the following holds.

1. $c(1,1,...,1) = 1$
2. If $x_i=0$ for any i then $c(x_1,x_2, \ldots, x_n)=0$.
3. $c(x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots,x_n)= c(x_1,x_2,\ldots,x_j,\ldots,x_i,\ldots,x_n)$
4. If $x_1=y_1, x_2=y_2, \ldots, x_{i-1}=y_{i-1}, x_{i+1}=y_{i+1}, \ldots, x_n=y_n$ are not equal zero,

and $x_i>y_i$ then $c(x_1,x_2,\ldots,x_n)>c(y_1,y_2,\ldots,y_n)$

4. $c(x,x,\ldots,x)=x$

In this case, associativity is not included, because it is not compatible with other desirable properties. Thus, it is necessary to define this operator over $[0,1]^n$.
Properties 1, 2 guaranties that the restriction of c to {0,1} matches the conjunction of the binary logic; property 3 is symmetry, but extended to the n-dimensional case.

The idempotent property is a necessary condition to preserve the significance of the given truth values, as well as the strict monotony of the operator for all the variables, whenever these are not zero. The latter assures the so-called sensibility of the connective predicates without losing the veto restrictions furnished by condition 2.

From 4, it follows that the operator is strictly increasing on $[0,1]^n$ for all variables. This assures the 'sensibility of the conjunctive predicates' to changes exerted on basic predicates. Indeed, property 4 determines the impossibility of associativity. This arises because there are no average strictly-increasing operators (Dubois/Prade, 1985), and c is an average operator, as proved in what follows. Let $(x_1,x_2,\ldots,x_n)$ in $[0,1]^n$, $x_m=\min\{x_i]$ y $x_M=\max\{x_i\}$; then, strictly increasing property verifies if

$$c(x_m, x_m, \ldots, x_m) \leq c(x_1, x_2, \ldots, x_n) \leq c(x_M, x_M, \ldots, x_M)$$

and from idempotent property

$$x_m \leq c(x_1, x_2, \ldots, x_n) \leq x_M$$

c is therefore and average operator.

Among the operators found in literature, the unique operator satisfying the previous axioms is the geometric mean.

$$c(x_1, x_2, ..., x_n) = (x_1.x_2.....x_n)^{1/n}$$

The following axioms for negation n: [0,1] → [0,1] are demanded whenever they preserve their convenience for the pursued goals:

6. n(0)=1
7. n(1)=0
8. n is continuous and strictly increasing.
9. n(n(x))=x

When 10. n(1/2) = 1/2, is added, which is a rather useful property to preserve a categorical meaning, it can be shown that: n(x) = 1-x.

On the other hand, disjunction must be defined in a sense that satisfies the De Morgan's Laws:

11. $n(c(x_1, x_2, ..., x_n))=d(n(x_1),n(x_2), ..., n(x_n))$
$n(d(x_1, x_2, ..., x_n))=c(n(x_1),n(x_2), ..., n(x_n))$

Thus, by taking the geometric mean as a definition for conjunction, disjunction is defined by

$$d(x_1, x_2, ..., x_n) = 1 - ((1 - x_1)(1 - x_2)....(1 - x_n))^{1/n}$$

From these properties, similar properties for disjunction can be shown. This is,

12. If x_i=1 for any i then $d(x_1, x_2, ..., x_n) = 1$
13. $d(x_1, x_2, ..., x_i, ..., x_j, ..., x_n)= d(x_1, x_2, ..., x_j, ..., x_i, ..., x_n)$
14. If $x_1 = y_1, x_2 = y_2, ..., x_{i-1} = y_{i-1}, x_{i+1} = y_{i+1}, ..., x_n = y_n$ are not equal zero,
and $x_i > y_i$, then $d(x_1, x_2, ..., x_n) > d(y_1, y_2, ..., y_n)$
15. d(x, x,..., x) = x

Other very important properties with utility in Decision Making situations are:

16. $|x - c(c(x, y), z))| \geq |x - c(x, y, z)|$
17. $|x - d(d(x, y), z))| \geq |x - d(x, y, z)|$

This elementary properties of average operators are very important because establishes that the influence of predicates in superior levels in a logical definition trip is more than the inferior levels.

The implication could be defined naturally as i(x,y)=d(n(x),y) like was made in previous papers (Espin, 2002)

For this implication the following properties hold
a) i(x, y) = 0 if and only if x = 1 and y = 0
b) i(x, y) = 1 if and only if x=0 or y=1
c) i(x, y) = i(n(y), n(x))
d) If $x_1, x_2, y \in \left]0,1\right[$ and
$x_1 < x_2$ *then* $i(x_1, y) > i(x_2, y)$
e) If $x, y_1, y_2 \in \left]0,1\right[$
and $y_1 < y_2$ *then* $i(x, y_1) < i(x, y_2)$

Such properties guarantee the extensionality of the binary logic implication, and provide an increasing implication over y for this logic system; but decreasing on x.

But this last characteristic it could be good for modelling deductions, but it have practical troubles in decision making problems, like it will be illustrated in this paper.

The naturality of this operator is an illusion; because the idea of its definition is influenced of the third excluded axiom. It would be better the use of:
i(x,y)=d(n(x),c(x,y)) a generalization of the so-called Zadeh implication.

This operator has the following very recommended properties:

17. i(x, y) = 0 if and only if x = 1 and y = 0
18. i(x, y) = 1 if and only if x=0 or (y=1 and x=1)
19. If $x, y_1, y_2 \in \left]0,1\right]$
and $y_1 < y_2$ *then* $i(x, y_1) < i(x, y_2)$

20. If $y \in \left]0,0.5\right[$ and
$x_1 < x_2$ *then* $i(x_1, y) > i(x_2, y)$

21. If $y \in [0.5,1]$.
There are $\alpha(y)$ *such that*
a) if $x_1, x_2 \in [0, \alpha(y)]$ *and* $x_1 < x_2$ *then* $i(x_1, y) > i(x_2, y)$
b) if $x_1, x_2 \in [\alpha(y), 1]$ *and* $x_1 < x_2$ *then* $i(x_1, y) < i(x_2, y)$

b) if $x_1, x_2 \in [\alpha(y), 1]$ *and* $x_1 < x_2$ *then* $i(x_1, y) < i(x_2, y)$

Existential and Universal quantifiers are defined in the case of limited sets on R^n, in natural way from the concepts of disjunction and conjunction, respectively.

$$\forall x\, p(x) = \begin{cases} e^{\frac{\int_X \ln(p(x))dx}{\int_X dx}} & if\ p(x) > 0\ for\ all\ x \in X \\ 0 & another\ case \end{cases}$$

$$\exists x\, p(x) = \begin{cases} 1 - e^{\frac{\int_X \ln(1-p(x))dx}{\int_X dx}} & if\ p(x) > 0\ for\ all\ x \in X \\ 1 & another\ case \end{cases}$$

A predicate such as $C(j) = \forall i\ (I_i \to P_{ji})$ where I_i is the predicate that models the sentence 'the goal i is important', and predicate P_{ji} expresses the accomplishment of the goal i aims or desires of i, , by the alternative j is a very general scheme for Decision Making problems. Note that, when predicates associated to attributes are considered, then the resulting situation can be identified as the classical multi-criterion decision problem. The very difficult form to obtain rigorously the so-called weights in normative additive models of decision making (French, 1986) is substituted hear for the corresponding truth values of the predicates 'The attribute i is important' for each attribute i.

Further, this approach guaranties the effective combination of 'intangibles', such as those obtained by consulting experts –considering category scales–, with quantitative information obtained by predicated that depend on the involved elements. The importance of the predicates, or attributes, can be obtained by a structurally complex way born from knowledge problem decision. This has been the case in some of the models developed by GEMINIS group to solve frequent managerial problems (Espin 1999, Espin 2002).

To summarize, this approach involves 'trade offs' between attributes, and the possibility of veto conditions for each of them; moreover, it permits to model the preferential independence by using conditional predicates. An empirical research program for the validation of this approach was carried out by GEMINIS group, which consisted of comparing a variety of management models (Espin,2002). In these studies, the best results were obtained by using Compensatory Logic, thus ratifying its convenience for selection, appraising, ranking, and classification problems.

The formulas of Propositions Compensatory Calculus are composed functions of the operators c,d, n and i. Following the spirit of the introduced Compensatory Predicates Calculus, a formula $f : [0,1]^n \to [0,1]$ of this Propositional Calculus would be true if f(x)>0 for every element of the domain, and

$$e^{\frac{\int_X \ln(f(x))dx}{\int_X dx}} > \frac{1}{2}\ .$$

In this sense the satisfied properties are exactly the formulas of Classic Bivalent Propositional Calculus using the natural implication or the Zadeh Generalized Implication.

Authors calculated the truth value of all the formulas of Kleene Axiomatic System using the 6.5 version of MATLAB. Axioms and Results of the calculus are the followings:

$$AX1:\ A \to (B \to A)$$
$$AX2:\ (A \to B) \to ((A \to (B \to C)) \to (A \to C))$$
$$AX3:\ A \to (B \to A \wedge B)$$
$$AX4:\ A \wedge B \to A \quad A \wedge B \to B$$
$$AX5:\ A \to A \vee B \quad B \to A \vee B$$
$$AX6:\ (A \to C) \to ((B \to C)) \to (A \vee B \to C))$$
$$AX7:\ (A \to B) \to ((A \to \neg B) \to \neg A))$$
$$AX8:\ \neg(\neg A) \to A$$

	Natural	Zadeh
Ax 1	0.5859	0.5685
Ax 2	0.5122	0.5073
Ax 3	0.5556	0.5669
Ax 4	0.5859	0.5661
Ax 5	0.8533	0.5859
Ax 6	0.5026	0.5038
Ax 7	0.5315	0.5137
Ax 8	0.5981	0.5981

ILLUSTRATION EXAMPLE

An hypothetical case of Multi attribute Decision Making is presented below to illustrate the convenience of Generalized Zadeh Implication in Compensatory Logic.
Fourth attributes are considered. The Importance Attributes Matrix and the Column Vectors of two alternatives are in the Matrixes I and P.

$$I = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \\ 0.5 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0.8 \\ 0.4 & 0.9 \\ 1 & 0.8 \\ 1. & 0.8 \end{bmatrix}$$

The alternative of the second column of matrix P is obviously better alternative than another one, because of the very bad performance of the first alternative in the second attribute, the more important one.

The Matrix T_1 of the truth values of each implications and the result matrix C_1 using natural implication are:

$$T_1 = \begin{bmatrix} 1.0000 & 0.6838 \\ 0.2254 & 0.6838 \\ 1.0000 & 0.6838 \\ 1.0000 & 0.6838 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.6890 & 0.6838 \end{bmatrix}$$

The Corresponding Matrixes for second alternative are:

$$T_2 = \begin{bmatrix} 0.6173 & 0.5713 \\ 0.3937 & 0.7735 \\ 0.6173 & 0.5713 \\ 0.6173 & 0.5713 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0.5517 & 0.6163 \end{bmatrix}$$

The first alternative is preferred by the first model, and the second one by the model using Generalized Zadeh Implication.

This express a wrong performance of the Model with Natural Implication, its incapacity of appreciate adequately the attributes importance, because this implication takes the constant value one whenever the truth value of consequent is 1, independently of the attribute importance.

4. CONCLUDING REMARKS

Decision Making problems are solved frequently in the framework of Fuzzy Logic using aggregation operators; but Decision Making needs to be hold using systems Probabilistic Logic and other multi-valued logic models are convenient for selection and ranking problems. Fuzzy Logic 'in a narrow sense' is convenient for appraising and classification problems. Definitions of implication may be stated in different ways, and their convenience is prescribed by the properties of the connectives. Compensatory Logic is convenient for selection, ranking, appraising, and classification problems. These qualities were exhibited through empirical research, in which a variety of Logical Systems were developed to resolve managerial problems. Demonstration of Traditional Propositions Calculus from this perspective is a good support to this axiomatic model. Demonstration of all the classical Predicates Calculus system from this, is being demonstrated.

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