A NEW ALGORITHM FOR ADAPTIVE MODELING OF MECHANICAL SYSTEMS

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ABSTRACT

The paper presents an application of Adaptive Matrix Filter method [1] to the modeling of the mechanical systems utilizing a new algorithm [2] for solving any combination of linear-nonlinear systems of equations. This algothm is based on the separation of linear equations in terms of some selected variables from the nonlinear ones. The linear group is solved by means of any method suitable for the linear system. This operation needs no iteration. The nonlinear group, however, is solved by an iterative technique based on a new formula developed using the Taylor series expansion. The method has successfully been applied to several examples of analytical systems as well as in some engineering applications with very good results. The proposed method needs the initial guess for nonlinear variables only. This is far less than needed in Newton-Raphson method. The method also has a very good convergence rate and it is shown that the results are not sensitive to the selected initial guess. The proposed method is most beneficial for Adaptive Modeling problems that very often involve large number of linear equations with limited number of nonlinear equations. This approach makes the Adaptive Matrix Filter method an effective tool in engineering applications

INTRODUCTION

The most basic step in performing the computer simulation and filtering of the data obtained from physical experiments is the selection of the model itself. Very often the model is based on incomplete empirical data. Selection of an inadequate model and parameters, that characterize it, can be the most important cause of errors of the computer simulation. A systematic approach for selection of appropriate models from a well-defined class of models could be very beneficial for the process. Using this approach we can correct the model as well as eliminate the noise and systematic errors from the measurement data.

Any data obtained from measurements as for example, displacements, strains or temperature carry some experimental errors due to inherent inaccuracies and deficiencies in the experimental techniques and measuring devices used. However, the quantities being measured must obey some laws of physic. In the cases involving Thermodynamics and Structural Analysis, these laws represent the equations of motion of thermo-elastic material and the equations of heat transfer. The quantities measured with errors do not satisfy the required model equations. However, this measured set of data may be enhanced substantially by determining a new set satisfying the model equations and be close as much as possible to the measured set. The transition from the measured set containing the experimental errors and noise, to the enhanced, corrected set is referred to as filtering.

Filtering and enhancing techniques for the analysis of the results of the numerical calculations and experimental data often use a set of models The proposed techniques and filters are based on the deterministic approach called Adaptive Matrix Filter (AMF). The algorithm may be achieved using the mathematical optimization technique in which the distance norm between the measured and calculated experimental data is selected as the objective function and then minimized subject to the equality constrained to represent the state equations.

The identification is performed on the basis of observations of the system response. An effective approach to detection of these parameters of structures which affect their thermal behavior can be described as follows: The change of the temperature, thermal strains and displacements are measured at the surface of the body in space and time. The recently developed photo-cameras for infrared photography make possible very precise detection of the temperature changes. It is also possible to measure the fields of the displacement using laser devices. The direct response of the system is used as the source of information. The heat conduction equations, thermo-elasticity equations and equations of motion (elasto-dynamic equations) can be used as model equations.

METHOD OF SOLUTION

The Adaptive Filter Matrix Method was detailed described in [1,2]. Here some short introduction to the method is included in the Example 2 In general the problems in the area of Adaptive Modeling require the solution of large systems of linear and nonlinear equations. Very often the solution that is based on the Finite Element Method is accompanied by a system of nonlinear equations. In this case the whole system becomes nonlinear and is solved using methods for the solution of non-linear equations. The Newton-Raphson iterations [3,4,5] is the method most commonly used. This method needs the calculation of first derivatives and the Jacobian matrix for the system. The solution is obtained by means of consecutive iterations. If the functions are differentiable with respect to the variables and behave well it is possible to find the solution in reasonable number of iterations. However, this needs the initial guess for all the variables taken sufficiently close to the simultaneous roots of the nonlinear system. This approach is not effective if the number of equations is large. There are problems with the convergence to correct the solution and problems with the initial guesses for the variables [3,4,5].

This paper presents a new method for the solution of a system of ***m+n*** nonlinear equations when the system of equations can be presented as two groups of equations. The first group of ***m*** equations is linear with respect to the selected ***m*** variables; the second group of ***n*** equations is nonlinear. The solution for the first group does not require any iterative procedures and can be found by means of any method for the system of linear equations. The proposed method uses iterations only for the nonlinear part and needs therefore fewer number of initial guesses as compared to those needed in Newton-Raphson method. The general system of equations can be presented in the following form:

$$\mathbf{f}(x,t) = 0,$$

$$\boldsymbol{\varphi}(x,t) = 0,$$

where the system $\boldsymbol{f_i}$ is linear with respect to the variables $\boldsymbol{x_i}$ with the assumption that the values of the variables $\boldsymbol{t_i}$ are known. Equations φ_n are non-linear with respect to the variables $\boldsymbol{x_i}$ and $\mathbf{t_i}$. Suppose that the vector ***t*** is the initial guess solution to the nonlinear variables of the system of the equations. Similarly, the vector *x* is the vector of initial solution for the linear part of the system equations based on using *t*. The vector *x* can be found by means of any method for the system of linear equations. Let $x+\Delta x$ and $t+\Delta t$ be a better approximate solution. Representing the functions ***f*** and $\boldsymbol{\varphi}$ by Taylor expansion in vector notation, we have

$$\boldsymbol{\varphi}(x,t) + \frac{\partial \boldsymbol{\varphi}}{\partial x}\Delta x + \frac{\partial \boldsymbol{\varphi}}{\partial t}\Delta t = 0 \qquad (1)$$

$$\mathbf{f}(x,t) + \frac{\partial \mathbf{f}}{\partial x}\Delta x + \frac{\partial \mathbf{f}}{\partial t}\Delta t = 0 \qquad (2)$$

Solving equations (1) and (2) with respect to Δx and Δt gives:

$$\Delta\mathbf{t} = -\left[I - \left(\frac{\partial \boldsymbol{\varphi}}{\partial t}\right)^{-1}\left(\frac{\partial \boldsymbol{\varphi}}{\partial x}\right)\left(\frac{\partial \mathbf{f}}{\partial x}\right)^{-1}\left(\frac{\partial \mathbf{f}}{\partial t}\right)\right]^{-1} \left[\frac{\partial \boldsymbol{\varphi}}{\partial t}\right]^{-1}\left[\boldsymbol{\varphi}(x,t)\right] \qquad (3)$$

and

$$\Delta\mathbf{x} = -\left[\frac{\partial \mathbf{f}}{\partial x}\right]^{-1}\left[\mathbf{f}(x,t) + \frac{\partial \mathbf{f}}{\partial t}\Delta t\right].$$

The new values of **t** and **x** are calculated as

$$\mathbf{x_{i+1}} = \mathbf{x_i} + \Delta\mathbf{x_i}, \quad \mathbf{t_{i+1}} = \mathbf{t_i} + \Delta\mathbf{t_i}$$

Below the following simple example is presented to explain the new method.

Example 1

Let us consider the following system of equations. The variables to be found are *x*, *y*, *z* and *t*.

$$tx + 2y - z = 21,$$

$$-2x + 3y - z = 0,$$

$$tx^2 - 2xy + y^2 - tz^2 = -213.$$

(4)

$$2x - y + 5z = 26,$$

The first and last equations are clearly nonlinear. However, following the procedure explained in the previous section, if we consider *t* as the variable to be found by iterations, then the first three equations will be linear in terms of *x*, *y* and *z* and can be solved for any given ***t***.

$$\mathbf{X}=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t & 2 & -1 \\ 2 & -1 & 5 \\ -2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 21 \\ 26 \\ 0 \end{bmatrix}. \qquad (5)$$

Table 1: The results for **Example** 1.

Table 1 also provides the record of the number of iterations and corresponding residue of ϕ_1 for different initial guesses t_1. The method found three independent solutions for x, y, z, and t. The most important fact is that it is almost independent on the initial value of the variable t and converges quickly to the solution. The number of iterations was very low, between 1 and 8.

The method was tested on many other examples with larger number of equations and nonlinear unknowns. It was found that the algorithm (3) was effective in all the cases, it converged fast and was able to find many solutions of for *t*. More detailed information about the method can be found in paper [3].

Initial guess for *t*	***x***	***y***	***z***	***t***	No. of iterations	ϕ_1 *(x,y,z,t)* error
0.5	15.6300	10.7900	1.1036	0.0331	6	-0.00004
1	-10.6594	-4.2339	8.6170	-3.5729	8	-0.00013
1.5	2.0000	3.0000	5.0000	9.9999	5	0.0017
2	2.0000	3.0000	5.0000	10.0000	4	0.0035
5	2.0000	3.0000	5.0000	10.0000	3	0.001
6	2.0000	3.0000	5.0000	10.0000	3	0.00007
7	2.0000	3.0000	5.0000	10.0000	3	0.000002
8	2.0001	3.0001	5.0000	9.9992	3	0.0248
9	2.0001	3.0001	5.0000	9.9993	2	0.0232
9.9	2.0000	3.0000	5.0000	10.0000	2	0.0003
10	2.0000	3.0000	5.0000	10.0000	1	0
50	2.0000	3.0000	5.0000	9.9999	3	0.0034
100	2.0000	3.0000	5.0000	9.9998	3	0.0068
1000	2.0001	3.0000	5.0000	9.9996	3	0.0117
10000	2.0001	3.0000	5.0000	9.9996	3	0.0124

The solution of the system of equations only needs the initial guess of t. Table 1 presents the solutions obtained for *x*, *y*, *z* and *t* using equation (12). The results are based on using $\delta = 0.1$ The required derivatives of the equations can be calculated as follows:

$$\Delta t = -[1-(\frac{1}{x^2-z^2})\,[2tx\text{-}2y \quad \text{-}2x\text{+}2y \quad \text{-}2tz] \begin{bmatrix} t & 2 & -1 \\ 2 & -1 & 5 \\ -2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}]^{-1} (\frac{1}{x^2-z^2}) \qquad (6)$$

$$(tx^2 - 2xy + y^2 - tz^2 + 213).$$

Example 2

In order to demonstrate the application of the proposed method in the field of adaptive modeling, let us consider the following case. A steel beam simply supported on two end bearings is under pure bending. The lateral deflection at 9 equally spaced nodes along the beam length has been measured. The Young's modulus of the beam is to be found using these measurements. However, this measured set of data may be enhanced substantially by determining a new set satisfying the model equations and be close as much as possible to the measured set. The transition from the measured set containing the experimental to the enhanced and corrected set is referred to as filtering. This technique has been fully explained in [1,2]. A Brief account of the method is given here.

Let us assume that $u_i^*(x_i)$ is the vector of measured lateral deflection of the beam at nine nodes that contain errors, *i* is the number of total measurements (*i*=9). The additional information about the system is presented in the form:

$$D \equiv EI\frac{d^2u}{dx^2} + M_0 = 0 \qquad (7)$$

$$P_1 \equiv u_1 = 0 \qquad (8)$$

$$P_2 \equiv u_9 = 0 \qquad (9)$$

E, I and M_0 are Young's modulus, moment of inertia and applied bending moment respectively. Vector u_i represents the corrected values of u_i^*. u_1 and u_9 are corrected deflections at left and right bearings respectively. By using the method of least square with Lagrange multipliers, the global error *R* in the interval of interest can be defined.

The derivatives of *R* with respect to u_i, η_j where μ_k η_j and μ_k are the Lagrange multipliers must be zero.. The finite difference representation of the differential operator is used for seven internal nodes. It can be shown [6] that the set of derivative equations can be presented in the matrix form:

$$\mathbf{IU} + \mathbf{N\lambda} - \mathbf{U}^* = 0, \qquad (10)$$

$$\mathbf{N^T U} - \mathbf{F}^* = 0 \qquad (11)$$

where U and U^* are the vectors of corrected and measured variables respectively. *I* is the unit, diagonal matrix with the order of 9. λ is the vector of Lagrange multipliers,

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_j \\ \eta_k \end{bmatrix}. \qquad (14)$$

Considering this matter of fact that *E* is unknown, the derivative of *R* with respect to *E* should also be zero. This **N** is the system matrix and F^* represents the loads. Superscript **T** denotes the transpose of matrix. In this case study, **N** is a constant matrix while F^* is a nonlinear function of E, the Young's modulus of the beam.

Equations (10) and (11) represent 18 linear equations in terms of 9 corrected node deflections, u_i (*i*=1-9) and 9 Lagrange multipliers λ_j (j=1-9). These equations, however, are nonlinear in terms of the unknown E, the Young's modulus of the beam.

In order to follow the same procedure as explained in examples1 to 3, equations (10) and (11) can be written in the following form that is more similar to previously explained notations:

$$f(u, \lambda, E) = \mathbf{A}\,\mathbf{X} - \mathbf{B} = 0, \qquad (12)$$

where:

$\mathbf{A}=\begin{bmatrix} I & N \\ N^T & 0 \end{bmatrix}$, $\mathbf{X}=[u_1\ u_2 \ldots u_9\ \lambda_1\ \lambda_2\ \ldots\ \lambda_9]^T$ and $\mathbf{B}=\begin{bmatrix} \mathbf{U}^* \\ \mathbf{F}^* \end{bmatrix}$.

The vector **X** represents the linear part of variables; the nonlinear part of variables consists only of E, the unknown Young's modulus of the beam.

For any given value of E, the matrix representation provides the unique solution for $\boldsymbol{\lambda}$ and **U** as:

$$\boldsymbol{\lambda} = (\mathbf{N^T N})^{-1}(\mathbf{N^T U}^* - \mathbf{F}^*),$$

$$\mathbf{U} = (\mathbf{I} - \mathbf{N}(\mathbf{N^T N})^{-1}\mathbf{N^T})\mathbf{U}^* + \mathbf{N}(\mathbf{N^T N})^{-1}\mathbf{F}^*,$$

The matrix **Filter** = $(\mathbf{I} - \mathbf{N}(\mathbf{N^T N})^{-1}\mathbf{N^T})$ is referred to as the filter matrix [6]. Considering that E is also unknown the derivative of R with respect to E should be zero. This leads to a nonlinear equation in the following form:

$$\varphi_1(\mathbf{u}, \boldsymbol{\lambda}, E) = \frac{\partial \mathbf{R}}{\partial E} = (\mathbf{u} - \mathbf{u}^*)^{\mathbf{T}} \frac{\partial \mathbf{u}}{\partial E} + \frac{\partial \boldsymbol{\lambda}^{\mathbf{T}}}{\partial E}(\mathbf{N^T u} - \mathbf{F}^*) + \boldsymbol{\lambda}^{\mathbf{T}}(\mathbf{N^T}\frac{\partial \mathbf{u}}{\partial E} - \frac{\partial \mathbf{F}^*}{\partial E}) = \mathbf{0} \qquad (15)$$

The system of equations in this example consists of 19 equations totally with 19 unknown (9 corrected deflections at 9 nodes, 9 Lagrange multipliers and Young's modulus of the beam). The first 18 equations represented by equation (22) are linear in terms of u_i (*i*=1-9) and λ_j (*j*=1-9) if the Young's modulus is given a certain value. This set can be solved without any iteration. The only nonlinear equation is equation (15). The vector of nonlinear variables consists only of one variable, *E* that is to be found by iteration. Solution of the system of equations with the suggested method only needs the initial guess for *E*.

Table 2: Calculated Young Modulus for the steel beam under pure bending.

Substituting the corresponding derivatives as follows in equation (3) gives the equations for E increment (for $t{=}E$).

Initial guess for E *(GPA)*	Calculated E *(GPA)*	No. of Iteration	Residue of ϕ_1 (10^{-7})
0.001	201.24	65	-9.5
1	201.23	34	-9.8
5	201.31	27	-8.3
10	201.35	24	-7.4
100	201.45	13	-5.5
195	201.27	8	-9.1
200	201.26	5	-9.2
330	201.28	18	-8.4

$$\frac{\partial \mathbf{f}}{\partial X} = \mathbf{A}, \quad \frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \frac{\partial \mathbf{f}}{\partial E} = -\frac{\partial \mathbf{B}}{\partial E}, \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} = \frac{\partial \boldsymbol{\varphi}_1}{\partial \mathbf{X}},$$

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} = \frac{\partial \phi_1}{\partial E},$$

A steel beam with a length of 600 *mm*, a width of 50 *mm* and a thickness of 5 *mm* has been considered. The pure applied bending moment was assumed as 10000 *N mm*. The theoretical lateral deflections at 9 equally spaced nodes along the beam length were used as measured values using a value of 200 *GPA* as Young Modulus with measurements error of the order of 5%. The expected value for E was 200*G*Pa.

Table 2 presents the calculated E for different initial guess when a value of 0.00001 is used for δ. The calculated values for corrected deflections are almost identical with the theoretical values. This is due to the fact that the filtering matrix is intelligent and recognizes correct data.

Table 4 shows that regardless of the initial guess for E, the program converges with good accuracy to the expected value of 200 *GPA*. The number of iterations is very small which indicates good convergence rate of the method. The range of initial guess in which the program converges is also very wide. The small difference between the correct value and the obtained results can be attributed to the application of only 9 elements to the solution of the problem.

Example 3

Rod pumping is the oldest and still the most common method of artificial lift used extensively in the oil well industry [8]. In this example an adaptive filter method has been used to model the dynamic behavior of the sucker rod string. The main concept was to replace the solution of the exact mathematical model of the pumping system by a simple model resulting in matrix operation in which the bottom-hole values are obtained as the product of the vector of the data at the top of the well and by a matrix of the system. Using this technique, the calculations of the bottom-hole values can be performed very fast and in a very simple way. These calculations would be easy to implement in one microprocessor of the computer. In the example presented here we found that to create the system matrix it is enough to replace the real system by a two-segment rod with appropriate dimensions. The simplified model was solved using D'Alembert's method [8]. The suggested technique uses the field dynamometer data at the polished rod and the calculated force and displacement at the plunger end from the analysis of the actual model for the same data. Then, it found the parameters of the equivalent two-segment rod solving the set of linear-nonlinear set of equations. Using the equivalent model the system matrix was created. The equivalent model was a very simple one, however it could replace the actual multi-segment actual rod.

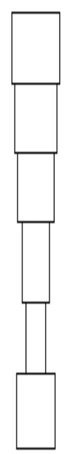

Fig. 1 A typical telescopic sucker rod string with 6 segments.

The data at the top of the sucker rod string were collected. The force and displacement at the polished rod were measured using a dynamometer. Then these data were used as the boundary conditions for the calculation of forces and displacements at the bottom of the rod of the well to define the conditions of the pump, effectiveness of pumping, production rate, etc.

The governing equation (1), for one dimensional motion of i'th segment of the rod, $u_i(x,t)$, is

$$a^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial t^2} + b_i \frac{\partial u_i}{\partial t}, \qquad \text{where } a^2 = \frac{E}{\rho},$$

$$b_i = \frac{\eta}{EA_i}.$$

E and ρ are Young's Modulus and density of the rod material respectively. A_i is the cross section of i'th segment of the rod and η is the damping per unit length.

The Adaptive Matrix Filter method was able to find successfully the geometric parameters of a simple equivalent model of 2-segment rod that has the same displacement and force distribution at the plunger as those of the actual six-segment rod when same input data at the polished rod are applied to both systems. The equivalent model, can be used to estimate the load and displacement at the plunger rod for any other dynamometer readings. This leads to shorter and efficient calculations of the bottom-hole values without loosing accuracy from the point of engineering applications. The more complete description of the problem and its solution can be found in [8].

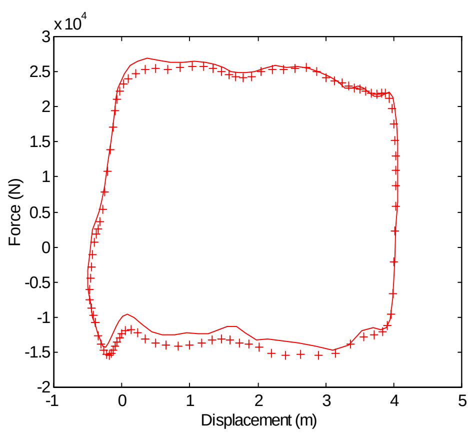

Fig. 2 Comparison between exact dynamic force at the plunger for the 6-segment rod (+) and its 2-segment equivalent model.

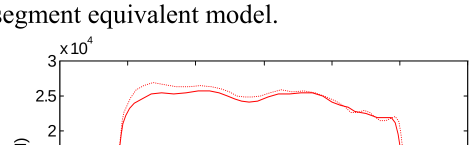

Fig. 3 Comparison between the exact dynamic force at the plunger and the force from the equivalent model for another set of dynamometer data.

CONCLUSIONS

The method of Adaptive Matrix Filter [1] cooperating together with the new method of the solution of equations of modeling and filtering problems consisting of a combination of the linear and nonlinear systems proved to be effective tool in the modeling the mechanical systems. The method has been successfully used in solving several different examples and other problems that are not presented here due to the lack of space. It has been shown that the method of the solution of linear – nonlinear equations is very effective. It converges fast and needs smaller number of initial guess values as compared with those needed in Newton-Raphson method. The method is most useful for solving engineering problems in which a large numbers of linear equations are coupled with a limited number of nonlinear equations. Application of Finite Element Method to the solution of the modeling problems results in large system of linear equations. Additional nonlinear constraints from the adaptive modeling provide the set of nonlinear equations.

REFERENCES

1. Lukasiewicz, S. A., Hojjati, M. H., Adaptive Matrix Filter, *Proceedings of CANCAM 2003 (19th)* University of Calgary, Calgary, Canada
2. Lukasiewicz.S.A,. Hojjatti M.H, A New Hybrid Method for Solving Linear-Nonlinear Systems of Equations. Submitted to the International Journal of Numerical Methods.
3. J.M. Ortega and W.C. Reinbolt, Iterative Solution of Nonlinear Equations in Several Variable, Academic Press, San Diego 1970.
4. K. A. Sikorski Optimal Solutions of Nonlinear Equations, Oxford University Press,
5. J. D. Faires, R.L.Burden , Numerical Methods (Book and disk with Instruction Manual) Brooks/Cole Prindle, Weber and Schmidt Series, 1993.
6. S. A. Lukasiewicz, Matrix Filter for Correcting Experimental Data, *Communications in Numerical Methods in Engineering*, Vol. 9, 797- 803 (1993).
7. Lukasiewicz, S. A., Hojjati, M. H., Adaptive Filter Matrix for Inverse Problem in Thermal Field, *Proceedings of 5th International Congress on Thermal Stress*, June 2003, Blacksburg, Virginia, USA.
8 M. H. Hojjati, S. A. Lukasiewicz, Adaptive Modeling of Sucker Rod String. Paper submitted to the JCPT, Journal of Canadian Petroleum

Technology,