APPLICATION OF QUEUEING THEORY TO REAL-TIME SYSTEMS WITH SHORTAGE OF MAINTENANCE TEAMS

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ABSTRACT

We present optimality conditions for real-time multiserver system with large number of identical servers and several non-identical channels working under maximum load regime with limited maintenance facilities. We calculate limiting values of system availability and its loss penalty function and show how to obtain optimal assignment probabilities which optimize these performance measures.

Keywords: assignment probabilities, availability, loss penalty, maintenance, real-time system

1. INTRODUCTION

Real-time systems (RTS) are defined as those for which correctness depends not only on the logical properties of the computed results, but also on the temporal properties of these results. In RTS an action performed too early/late, may be useless, and sometimes harmful. Examples include industrial automation, traffic control, robotics, intelligence and defense systems, etc.

There is a growing consensus that the use of analytical methods of queueing theory [2] together with modern computation approaches (such as Artificial Intelligence AI [9], Genetic algorithms [10], Neural Networks NN [5,12] and Evolutionary Computations EC [1]) and simulation techniques (such as Score Function [11] and Perturbation Analysis [3] could have significant benefits in developing RTS.

The particular interest in RTS with a zero deadline for the beginning of job processing was aroused by military intelligence problems involving unmanned air vehicles (UAV). Kreimer and Mehrez have proved [8] that the non-mix policy maximizes the availability of a multiserver single-channel RTS. In [6] and [7] Kreimer have applied birth-and-death processes in analysis of a multiserver RTS with different channels operating under a maximum load regime. Ianovsky and Kreimer [4] have obtained optimal assignment probabilities to maximize availability of RTS with two different channels. This paper considers the model developed in [7] and extends the results of [4] for arbitrary number of different channels.

Our purpose is to calculate limiting values of availability and loss penalty function for RTS with large number of servers, and then to obtain optimal assignment probabilities which optimize these performance measures.

2. DESCRIPTION OF THE SYSTEM

We consider a RTS with a zero deadline for the beginning of job processing. As a matter of fact most of monitoring RTS are of this type.
The most important characteristics of RTS are summarized in [8]. Jobs in RTS are executed *immediately* upon arrival, conditional on system availability. Storage of non-completed jobs or their parts is impossible. Nevertheless, queueing theory methodology can be successfully applied in analysis of RTS.

A RTS under consideration consists of N servers that provide service for the requests of real-time jobs, arriving via *r different* channels. The system works under a maximum load of nonstop data arrival. Thus there is exactly one request of real-time job in each channel at *any* instant, and therefore one server at most is used to process the job in the channel. The total number of working servers is at most *r* (as a number of channels). The *i*-th server can provide S_i time units of service before requiring *R* continuos time units of maintenance, after which it is again available, and so on. Both S_i (*i*=1,...,*N*) and *R* are independent exponentially distributed random values with parameters μ_i ($i=\overline{1,r}$) and λ respectively. Service of a job continues while there are available servers in the system. Different parts of the same job can be processed by different servers. Any part of the job that is not served immediately in real-time is lost.

Each channel has its own specifications and requires different kinds of service. Server, which was sent to maintenance, is assigned to the i-th channel with probability p_i ($i=\overline{1,r}$). It gets the appropriate kind of maintenance, and therefore cannot be sent to another channel. The duration R of maintenance work is exponentially distributed with parameter λ, and does not depend on the assigned channel. After maintenance, the server will either be on stand-by or serving the assigned channel.

It is assumed that there are only K ($K \le N$) maintenance teams. Thus a shortage of maintenance teams occurs when there are more than K servers out of order. Then the server will wait for repair in the queue.

3. THE PROBLEM AND ITS SOLUTION

Kreimer [7] has shown that steady state probabilities of the RTS under consideration are given by the following formulae:

$$p_{n_1,\ldots,n_r} = \begin{cases} K^{\sum_{i=1}^{r} n_i} \prod_{i=1}^{r} \rho_i^{n_i}\, p_{0,\ldots,0}, \; if \sum_{i=1}^{r} n_i \le N-K \\ \dfrac{K^{N-K}K!}{\left(N-\sum_{i=1}^{r} n_i\right)!} \prod_{i=1}^{r} \rho_i^{n_i}\, p_{0,\ldots,0}, \; if \sum_{i=1}^{r} n_i > N-K \end{cases} \quad (1)$$

and

$$p_{0,\ldots,0} = \left\{ \sum_{\sum_{i=1}^{r} n_i = 0}^{N-K} K^{\sum_{i=1}^{r} n_i} \prod_{i=1}^{r} \rho_i^{n_i} + \right.$$

$$\left. + \sum_{\sum_{i=1}^{r} n_i = N-K+1}^{N} \frac{K^{N-K}K!\prod_{i=1}^{r}\rho_i^{n_i}}{\left(N-\sum_{i=1}^{r} n_i\right)!} \right\}^{-1} \quad (2)$$

where n_i ($i=\overline{1,r}$) is a number of fixed servers assigned to the i-th channel, $p_{n_1,n_2,\ldots,n_r}$ the corresponding steady state probability, and $\rho_i = \lambda_i / \mu_i$ ($\lambda_i = \lambda p_i$).

For a *multichannel* system (r>1) operating under maximum load regime, the availability is given by the following formula (see [6]):

Av = E[number of busy servers]/ r. (3)

Taking into account equations (1)-(2), we obtain

$$Av_N(\rho_1,\ldots,\rho_r) = \sum_{k=1}^{r}\left(1-P_N^{(k)}(\rho_1,\ldots,\rho_r)\right) \Big/ r, \quad (4)$$

where $P_N^{(k)}(\rho_1,\ldots,\rho_r)$ is the probability that channel k ($k=\overline{1,r}$) is not served.

$$TC_N(\rho_1,\ldots,\rho_r) = \sum_{k=1}^{r} C_k P_N^{(k)}(\rho_1,\ldots,\rho_r) \quad (5)$$

is an average loss penalty cost, where C_k ($k=\overline{1,r}$) is the cost of the time unit during which the k-th channel is not served (penalty).

<u>*Assertion:*</u> $Av_N(\rho_1,\ldots,\rho_r) = 1 - TC_N(\rho_1,\ldots,\rho_r)/r$, for $C_i = 1$, $i=\overline{1,r}$.

Functions (4) and (5) can be maximized and minimized respectively for any finite value of N, but only numerically – analytical solutions are available. When $N \to \infty$, analytical expressions of availability and loss penalty function as well as optimal values for assignment probabilities can be obtained.

Let

$$S_N^r(\rho_1,\ldots,\rho_r) = \sum_{\substack{n_1,\ldots,n_r \ge 0 \\ n_1+\ldots+n_r \le N}} \frac{K^{\min\left(\sum_{i=1}^{r} n_i,\, N-K\right)} K!}{\min\left(N-\sum_{i=1}^{r} n_i,\, K\right)!} \prod_{j=1}^{r} \rho_j^{n_j} =$$

$$= \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \cdots \sum_{n_r=0}^{N-\sum_{i=1}^{r-1} n_i} \frac{K^{\min\left(\sum_{i=1}^{r} n_i,\, N-K\right)} K!}{\min\left(N-\sum_{i=1}^{r} n_i,\, K\right)!} \prod_{j=1}^{r} \rho_j^{n_j},$$

then

$$P_N^{(k)}(\rho_1,\ldots,\rho_r) = \frac{\sum_{\substack{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_r \ge 0 \\ n_1+\ldots+n_{k-1}+n_{k+1}+\ldots+n_r \le N}} \dfrac{p_{n_1,\ldots,n_{k-1},0,n_{k+1},\ldots,n_r}}{p_{0,\ldots,0}}}{\sum_{\substack{n_1,\ldots,n_r \ge 0 \\ n_1+\ldots+n_r \le N}} \dfrac{p_{n_1,\ldots,n_r}}{p_{0,\ldots,0}}} =$$

$$= \frac{\sum_{\substack{n_1,\ldots,n_{k-1},n_{k+1},\ldots,n_r \ge 0 \\ n_1+\ldots+n_{k-1}+n_{k+1}+\ldots+n_r \le N}} \dfrac{K^{\min\left(\sum_{i=1, i\ne k}^{r} n_i,\, N-K\right)} K!}{\min\left(N-\sum_{i=1, i\ne k}^{r} n_i,\, K\right)!} \prod_{j=1, j\ne k}^{r} \rho_j^{n_j}}{\sum_{\substack{n_1,\ldots,n_r \ge 0 \\ n_1+\ldots+n_r \le N}} \dfrac{K^{\min\left(\sum_{i=1}^{r} n_i,\, N-K\right)} K!}{\min\left(N-\sum_{i=1}^{r} n_i,\, K\right)!} \prod_{j=1}^{r} \rho_j^{n_j}} =$$

$$= \frac{S_N^{r-1}(\rho_1,\ldots,\rho_{k-1},\rho_{k+1},\ldots,\rho_r)}{S_N^r(\rho_1,\ldots,\rho_r)}. \quad (6)$$

Using (1), (6), and the following Lemmas and Corollaries will help us to obtain the main results.

<u>*Lemma 1:*</u>

$S_N^r(\rho_1,\ldots,\rho_r) = \left[\rho_i S_N^{r-1}(\rho_1,\ldots,\rho_{j-1},\rho_{j+1},\ldots,\rho_r) - \right.$

$$-\rho_j S_N^{r-1}(\rho_1,...,\rho_{i-1},\rho_{i+1},...,\rho_r)\Big]\frac{1}{\rho_i-\rho_j},$$

where $\rho_1,...,\rho_r>0$ and $\rho_i\neq\rho_j$ for $i,j=\overline{1,r}$, $i\neq j$, $r\geq 2$, $N\geq 1$.

Lemma 2: $$S_N^r(\rho_1,...,\rho_r)=\sum_{i=1}^{r}\frac{\rho_i^{r-1}S_N^1(\rho_i)}{\prod\limits_{\substack{k=1\\k\neq i}}^{r}(\rho_i-\rho_k)},$$

where $\rho_1,...,\rho_r>0$ and $\rho_i\neq\rho_j$ for $i,j=\overline{1,r}$, $i\neq j$, $r\geq 2$, $N\geq 1$.

Lemma 3: $$\sum_{i=1}^{r}\frac{\rho_i^{r-2}}{\prod\limits_{\substack{k=1\\k\neq i}}^{r}(\rho_i-\rho_k)}=0,$$

where $\rho_1,...,\rho_r>0$ and $\rho_i\neq\rho_j$ for $i,j=\overline{1,r}$, $i\neq j$, $r\geq 2$.

Corollary 1: $$\sum_{n=2}^{r}\frac{\rho_n^{r-3}}{\left(1-\frac{\rho_1}{\rho_n}\right)\prod\limits_{\substack{k=2\\k\neq n}}^{r}(\rho_n-\rho_k)}=-\frac{\rho_1^{r-2}}{\prod\limits_{k=2}^{r}(\rho_1-\rho_k)},$$

where $\rho_1,...,\rho_r>0$ and $\rho_i\neq\rho_j$ for $i,j=\overline{1,r}$, $i\neq j$, $r\geq 2$, $\prod\limits_{\substack{k=n\\k\neq n}}^{n}(\rho_n-\rho_k)=1$.

Lemma 4: $$\sum_{n_1=0}^{N}\sum_{n_2=0}^{N-n_1}\sum_{n_3=0}^{N-n_1-n_2}\ldots\sum_{n_i=0}^{N-\sum_{j=1}^{i-1}n_j}1=\binom{N+i}{i},\ i\geq 1.$$

Lemma 5: For $t\geq 1$ it is true

$$S_N^t(\rho_1,...,\rho_m,\rho_{m+1},...,\rho_t)=$$
$$=\begin{cases}\sum\limits_{i=0}^{N}\binom{i+m-1}{i}\frac{K^{\min(i,N-K)}K!}{\min(N-i,K)!}\rho^i, & m=t,\\ \sum\limits_{i=0}^{N}\binom{i+m-1}{i}(K\rho)^i S_{N-i}^1(\rho_t), & m=t-1,\\ \sum\limits_{i=0}^{N}\binom{i+m-1}{i}(K\rho)^i\sum\limits_{j=m+1}^{t}\frac{\rho_j^{t-m-1}S_{N-i}^1(\rho_j)}{\prod\limits_{\substack{n=m+1\\n\neq j}}^{t}(\rho_j-\rho_n)}, & m=\overline{1,t-2}.\end{cases}$$

Lemma 6: $$F(s,i,q)=\sum_{m=0}^{s}(m+1)(m+2)\cdot\ldots\cdot(m+i)q^m=$$
$$=\frac{i!}{(1-q)^{i+1}}-i!q^{s+1}\sum_{k=1}^{i+1}\frac{1}{(1-q)^k}\binom{s+i-k+1}{s},\qquad q\neq 1,$$
$i\geq 1$.

$$F(s,i,1)=\sum_{m=0}^{s}(m+1)(m+2)\cdot\ldots\cdot(m+i)=$$
$$=\frac{(s+1)\cdot\ldots\cdot(s+i+1)}{i+1},\ i\geq 1.$$

Lemma 7: $$\sum_{i=0}^{N}\binom{i+m-1}{i}(K\rho)^i S_{N-i}^1(\rho_j)=\frac{S_N^1(\rho_j)}{\left(1-\frac{\rho}{\rho_j}\right)^m}-$$
$$-\frac{\rho}{\rho_j}\sum_{e=1}^{m}\frac{1}{\left(1-\frac{\rho}{\rho_j}\right)^e}\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+m-e}{s}\rho^s,$$

where $0<\rho_j<\rho$, $m\geq 2$.

Lemma 8: $$S_N^t(\rho_1,...,\rho_m,\rho_{m+1},...,\rho_t)=$$
$$=\sum_{n=m+1}^{t}\frac{\rho_n^{t-1}S_N^1(\rho_n)}{(\rho_n-\rho)^m\prod\limits_{\substack{i=m+1\\i\neq n}}^{t}(\rho_n-\rho_i)}-$$
$$-\rho\sum_{e=1}^{m}\left(\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+m-e}{s}\rho^s\right)*$$
$$*\left(\sum_{n=m+1}^{t}\frac{\rho_n^{t-m-2}}{\left(1-\frac{\rho}{\rho_n}\right)^e\prod\limits_{\substack{i=m+1\\i\neq n}}^{t}(\rho_n-\rho_i)}\right),\ m=\overline{1,t-2}.$$

Corollary 2: Let $\rho_1=...=\rho_m=\rho$ ($m\in\{0,1,...,t\}$), $0<\rho_i<\rho$ and $\rho_i\neq\rho_j$ ($i,j=\overline{m+1,t}$, $i\neq j$), $\sum\limits_{k=1}^{0}=0$ and $\prod\limits_{\substack{k=t\\k\neq t}}^{t}(\rho_t-\rho_k)=1$. Then, for $t\geq 1$, by Lemmas 2, 5, 7, 8 we can conclude, that

$$S_N^t(\rho_1,...,\rho_m,\rho_{m+1},...,\rho_t)=$$
$$=\begin{cases}\sum\limits_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+m-1}{s}\rho^s, & \text{if } m=t,\\ \sum\limits_{n=m+1}^{t}\frac{\rho_n^{t-1}S_N^1(\rho_n)}{(\rho_n-\rho)^m\prod\limits_{\substack{i=m+1\\i\neq n}}^{t}(\rho_n-\rho_i)}- \\ -\rho\sum\limits_{e=1}^{m}\left(\sum\limits_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+m-e}{s}\rho^s\right)* \\ *\left(\sum\limits_{n=m+1}^{t}\frac{\rho_n^{t-m-2}}{\left(1-\frac{\rho}{\rho_n}\right)^e\prod\limits_{\substack{i=m+1\\i\neq n}}^{t}(\rho_n-\rho_i)}\right), & \text{if } m=\overline{0,t-1}.\end{cases}$$

Lemma 9:

$$\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+i}{s}\rho^s=\frac{1}{(1-K\rho)^{i+1}}+O(\alpha^N),$$

for $0<K\rho<\alpha<1$ and $i\ge 0$.

$$\frac{\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+i}{s}\rho^s}{\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+i+1}{s}\rho^s}=O\left(\frac{1}{N}\right),$$

for $K\rho\ge 1$ and $i\ge 0$.

Corollary 3: Lemma 9, when $K\rho\ge 1$, results in

$$\frac{\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+i}{s}\rho^s}{\sum_{s=0}^{N}\frac{K^{\min(s,N-K)}K!}{\min(N-s,K)!}\binom{s+i+j}{s}\rho^s}=O\left(\frac{1}{N^j}\right),\ i\ge 0,$$

$j\ge 1$.

Corollary 4: Let $\rho_1=...=\rho_m=\rho$ ($m\in\{0,1,...,t\}$, $t\in\{0,1,...,r\}$), $0<\rho_i<\rho$ ($i=\overline{m+1,t}$), $\rho_i=0$ ($i=\overline{t+1,r}$), $\sum_{k=1}^{0}=0$ and $\prod_{\substack{k=t\\k\neq t}}^{t}(\rho_t-\rho_k)=1$. Then, for $K\max_{i=\overline{1,r}}(\rho_i)<\alpha<1$ and $r\ge 1$, we have

$$S_N^r(\rho_1,...,\rho_m,\rho_{m+1},...,\rho_r)=\frac{1}{\prod_{j=1}^{r}(1-K\rho_j)}+O(\alpha^N).$$

Our main results can be formulated as follows:

Theorem 1:

$$P^{(i)}(\rho_1,...,\rho_r)=\lim_{N\to\infty}P_N^{(i)}(\rho_1,...,\rho_r)=1-\rho_i\min\left(K,\frac{1}{\rho}\right),$$

$i=\overline{1,r}$, where $\rho=\max_{i=\overline{1,r}}\rho_i$.

Proof:

For $K\rho\ge 1$, we consider two cases.

Case 1: If there is $n\in\{1,2,...,r\}$ such that $\rho_n\neq\rho$, then for any k and i ($i,k\in\{1,2,...,r\}$) such that $\rho_k\neq\rho$ and $\rho_i=\rho$ it can be written

$$P_N^{(k)}(\rho_1,...,\rho_r)\overset{(6)}{=}\frac{S_N^{r-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_r)}{S_N^r(\rho_1,...,\rho_r)}\overset{\text{Lemma1}}{=}$$

$$=\frac{\rho_k-\rho_i}{\rho_k\frac{S_N^{r-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_k,\rho_{i+1},...,\rho_r)}{S_N^{r-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_i,\rho_{i+1},...,\rho_r)}-\rho_i}\quad(7)$$

and

$$P_N^{(i)}(\rho_1,...,\rho_r)\overset{(6)}{=}\frac{S_N^{r-1}(\rho_1,...,\rho_{i-1},\rho_{i+1},...,\rho_r)}{S_N^r(\rho_1,...,\rho_r)}=$$

$$\overset{\text{by definition of } S_N^r(\cdot)}{=}\frac{S_N^r(\rho_1,...,\rho_{i-1},0,\rho_{i+1},...,\rho_r)}{S_N^r(\rho_1,...,\rho_r)}.\quad(8)$$

Case 2: If $\rho_1=...=\rho_r=\rho$, then

$$P_N^{(k)}(\rho_1,...,\rho_r)\overset{(6)}{=}\frac{S_N^{r-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_r)}{S_N^r(\rho_1,...,\rho_r)}=$$

$$\overset{\text{by definition of } S_N^r(\cdot)}{=}\frac{S_N^r(\rho_1,...,\rho_{k-1},0,\rho_{k+1},...,\rho_r)}{S_N^r(\rho_1,...,\rho_r)},\ k=\overline{1,r}.\quad(9)$$

Therefore, from (7), (8) and (9), it can be concluded that for $r\ge 2$ the following equality

$$\lim_{N\to\infty}\frac{S_N^{t-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_k,\rho_{i+1},...,\rho_t)}{S_N^{t-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_i,\rho_{i+1},...,\rho_t)}=0,\quad(10)$$

where $t\ge 2$, $\rho_i=\rho$, $\rho_k\neq\rho$, proves this Theorem, when $K\rho\ge 1$, in both cases.

Let m is a number of parameters ρ_j such that $\rho_j=\rho$ for the denominator of (10). Consequently, a number of parameters ρ_j such that $\rho_j=\rho$ for the numerator of (10) is m-1 and from (7), (8) and (9) we have that $m\in\{1,2,...,t-1\}$ for any case. Then

$$\frac{S_N^{t-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_k,\rho_{i+1},...,\rho_t)}{S_N^{t-1}(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_{i-1},\rho_i,\rho_{i+1},...,\rho_t)}=$$

$$=\frac{S_N^{t-1}(\rho_1^{(1)},...,\rho_{t-1}^{(1)})}{S_N^{t-1}(\rho_1^{(2)},...,\rho_{t-1}^{(2)})},\quad(11)$$

where $\rho_1^{(1)}=...=\rho_{m-1}^{(1)}=\rho_1^{(2)}=...=\rho_m^{(2)}=\rho$, $0\le\rho_i^{(1)}<\rho$ ($i=\overline{m,t-1}$), $0\le\rho_i^{(2)}<\rho$ ($i=\overline{m+1,t-1}$).

From (11) we have

$$\frac{S_N^{t-1}(\rho_1^{(1)},...,\rho_{t-1}^{(1)})}{S_N^{t-1}(\rho_1^{(2)},...,\rho_{t-1}^{(2)})}\overset{\text{by definition of } S_N^t(\cdot)}{\le}\frac{S_N^{t-1}(\rho_1^{(1)},...,\rho_{t-1}^{(1)})}{S_N^m(\rho,...,\rho)}<$$

$$<\frac{S_N^{t-1}(\rho_1^{(3)},...,\rho_{t-1}^{(3)})}{S_N^m(\rho,...,\rho)},\quad(12)$$

where $\rho_1^{(3)}=...=\rho_{m-1}^{(3)}=\rho$, $0\le\rho_i^{(1)}<\rho_i^{(3)}<\rho$ and $\rho_i^{(3)}\neq\rho_j^{(3)}$ for $i,j=\overline{m,t-1}$ and $i\neq j$.

Using the inequalities $\frac{\rho_i^{(3)}}{\rho}<1$ for $i=\overline{m,t-1}$, $K\rho_i^{(3)}<1$ for $K\rho=1$ and $i=\overline{m,t-1}$, and $\frac{1}{K\rho}<1$, when $K\rho>1$, we obtain for $i=\overline{m,t-1}$

$$\frac{S_N^1\left(\rho_i^{(3)}\right)}{S_N^m(\rho,\dots,\rho)} \overset{\text{Corollary 2}}{=} \frac{S_N^1\left(\rho_i^{(3)}\right)}{\sum_{s=0}^{N} \frac{K^{\min(s,N-K)} K!}{\min(N-s,K)!} \binom{s+m-1}{s} \rho^s} \le$$

$$\le \frac{S_N^1\left(\rho_i^{(3)}\right)}{S_N^1(\rho)} =$$

$$= \frac{\sum_{n=0}^{N-K}\left(K\rho_i^{(3)}\right)^n + K! \frac{\left(K\rho_i^{(3)}\right)^N}{K^K} \sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{1}{\rho_i^{(3)}}\right)^n}{\sum_{n=0}^{N-K}(K\rho)^n + K! \frac{(K\rho)^N}{K^K} \sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{1}{\rho}\right)^n} =$$

$$= \frac{\frac{1-\left(K\rho_i^{(3)}\right)^{N-K+1}}{1-K\rho_i^{(3)}} + K! \frac{\left(K\rho_i^{(3)}\right)^N}{K^K} \sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{1}{\rho_i^{(3)}}\right)^n}{\frac{1-(K\rho)^{N-K+1}}{1-K\rho} + K! \frac{(K\rho)^N}{K^K} \sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{1}{\rho}\right)^n} =$$

$$= \begin{cases} O(1/N), & if \;\; K\rho = 1 \\ O\left(\alpha_1^N\right), & if \;\; K\rho > 1 \end{cases}, \tag{13}$$

for any α_1 such that $\max\left\{\frac{1}{K\rho}, \max_{\substack{i=\overline{1,r} \\ \rho_i<\rho}} \frac{\rho_i}{\rho}\right\} < \alpha_1 < 1$.

From (13) and the Corollaries 2 and 3 we conclude that

$$\frac{S_N^{t-1}\left(\rho_1^{(3)},\dots,\rho_{t-1}^{(3)}\right)}{S_N^m(\rho,\dots,\rho)} =$$

$$= \begin{cases} O(1/N), & if \;\; K\rho = 1 \\ O\left(\alpha_1^N\right), & if \;\; K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \;\; K\rho > 1 \,\&\, m \in \{2,3,\dots,t-1\} \end{cases} \tag{14}$$

Thus, by (11), (12) and (14), the next equality is true

$$\frac{S_N^{t-1}(\rho_1,\dots,\rho_{k-1},\rho_{k+1},\dots,\rho_{i-1},\rho_k,\rho_{i+1},\dots,\rho_t)}{S_N^{t-1}(\rho_1,\dots,\rho_{k-1},\rho_{k+1},\dots,\rho_{i-1},\rho_i,\rho_{i+1},\dots,\rho_t)} =$$

$$= \begin{cases} O(1/N), & if \;\; K\rho = 1 \\ O\left(\alpha_1^N\right), & if \;\; K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \;\; K\rho > 1 \,\&\, m \in \{2,3,\dots,t-1\} \end{cases} \tag{15}$$

that proves (10).

When $K\rho < 1$, we have, by Corollary 4, that

$$P_N^{(j)}(\rho_1,\dots,\rho_r) \overset{(6)}{=} \frac{S_N^{r-1}(\rho_1,\dots,\rho_{j-1},\rho_{j+1},\dots,\rho_r)}{S_N^r(\rho_1,\dots,\rho_r)} =$$

$$= \frac{\frac{1}{\prod_{\substack{i=1\\i\ne j}}^{r}(1-K\rho_i)} + O\left(\alpha_2^N\right)}{\frac{1}{\prod_{i=1}^{r}(1-K\rho_i)} + O\left(\alpha_2^N\right)} = \frac{\frac{1}{\prod_{\substack{i=1\\i\ne j}}^{r}(1-K\rho_i)}}{\frac{1}{\prod_{i=1}^{r}(1-K\rho_i)}} + O\left(\alpha_2^N\right) =$$

$$= 1 - K\rho_j + O\left(\alpha_2^N\right), \tag{16}$$

for any α_2 such that $K\rho < \alpha_2 < 1$.

Formula (16) proves this Theorem when $K\rho < 1$.

Q.E.D.

Theorem 2:

$$TC(\rho_1,\dots,\rho_r) = \lim_{N\to\infty} TC_N(\rho_1,\dots,\rho_r) =$$

$$= \sum_{i=1}^{r} C_i\left(1 - \rho_i \min\left(K, \frac{1}{\rho}\right)\right),$$

$$Av(\rho_1,\dots,\rho_r) = \lim_{N\to\infty} Av_N(\rho_1,\dots,\rho_r) =$$

$$= \frac{\min\left(K, \frac{1}{\rho}\right)}{r} \sum_{i=1}^{r} \rho_i \;.$$

Proof:

This Theorem follows from (4), (5) and Theorem 1.

Q.E.D.

Theorem 3: The rate of convergence

$$P_N^{(i)}(\rho_1,\dots,\rho_r) \xrightarrow[N\to\infty]{} P^{(i)}(\rho_1,\dots,\rho_r)\;,\; i = \overline{1,r},$$

and therefore

$$TC_N(\rho_1,\dots,\rho_r) \xrightarrow[N\to\infty]{} TC(\rho_1,\dots,\rho_r)$$

and

$$Av_N(\rho_1,\dots,\rho_r) \xrightarrow[N\to\infty]{} Av(\rho_1,\dots,\rho_r)$$

is $\begin{cases} O\left(\alpha_1^N\right), & if \;\; K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \;\; K\rho > 1 \,\&\, m \in \{2,3,\dots,r\} \\ O(1/N), & if \;\; K\rho = 1 \\ O\left(\alpha_2^N\right), & if \;\; K\rho < 1 \end{cases}$,

where $\max\left\{\frac{1}{K\rho}, \max_{\substack{i=\overline{1,r} \\ \rho_i<\rho}} \frac{\rho_i}{\rho}\right\} < \alpha_1 < 1$, $K\rho < \alpha_2 < 1$,

and *m* is number of $\rho_i = \rho$, $i = \overline{1,r}$.

Proof:

For $K\rho \ge 1$, we consider the two cases.

<u>*Case 1:*</u> If there is $n \in \{1,2,\dots,r\}$ such that $\rho_n \ne \rho$, then for any *k* and *i* ($i,k \in \{1,2,\dots,r\}$) such that $\rho_k \ne \rho$ and $\rho_i = \rho$ it can be written

$$P_N^{(k)}(\rho_1,\dots,\rho_r) \overset{(7)\text{ and }(15)}{=}$$

$$= \frac{\rho_k - \rho_i}{\rho_k * \left.\begin{cases} O(1/N), & if \;\; K\rho = 1 \\ O\left(\alpha_1^N\right), & if \;\; K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \;\; K\rho > 1 \,\&\, m \in \{2,3,\dots,r-1\} \end{cases}\right| - \rho_i}$$

and

$$P_N^{(i)}(\rho_1,\dots,\rho_r) \overset{(8)\text{ and }(15)}{=}$$

$$= \begin{cases} O(1/N), & if \ K\rho = 1 \\ O(\alpha_1^N), & if \ K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \ K\rho > 1 \,\&\, m \in \{2, 3, \dots, r-1\} \end{cases}.$$

Therefore, by Theorem 1,

$$\left| P_N^{(k)}(\rho_1, \dots, \rho_r) - P^{(k)}(\rho_1, \dots, \rho_r) \right| = (\rho_i - \rho_k) *$$

$$* \left| \frac{1}{\rho_k * \begin{cases} O(1/N), & if \ K\rho = 1 \\ O(\alpha_1^N), & if \ K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \ K\rho > 1 \,\&\, m \in \{2, 3, \dots, r-1\} \end{cases} - \rho_i} + \right.$$

$$\left. + \frac{1}{\rho_i} \right| = \begin{cases} O(1/N), & if \ K\rho = 1 \\ O(\alpha_1^N), & if \ K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \ K\rho > 1 \,\&\, m \in \{2, 3, \dots, r-1\} \end{cases}, \quad (17)$$

and

$$\left| P_N^{(i)}(\rho_1, \dots, \rho_r) - P^{(i)}(\rho_1, \dots, \rho_r) \right| = \left| P_N^{(i)}(\rho_1, \dots, \rho_r) \right| =$$

$$= \begin{cases} O(1/N), & if \ K\rho = 1 \\ O(\alpha_1^N), & if \ K\rho > 1 \,\&\, m = 1 \\ O(1/N), & if \ K\rho > 1 \,\&\, m \in \{2, 3, \dots, r-1\} \end{cases}. \quad (18)$$

<u>*Case 2:*</u> If $\rho_1 = \dots = \rho_r = \rho$, then

$$P_N^{(k)}(\rho_1, \dots, \rho_r) \overset{(9) \text{ and } (15)}{=} O\left(\frac{1}{N}\right), \ k = \overline{1, r}.$$

Therefore, by Theorem 1,

$$\left| P_N^{(k)}(\rho_1, \dots, \rho_r) - P^{(k)}(\rho_1, \dots, \rho_r) \right| =$$

$$= \left| P_N^{(k)}(\rho_1, \dots, \rho_r) \right| = O\left(\frac{1}{N}\right), \ k = \overline{1, r}. \quad (19)$$

(17), (18) and (19) prove this Theorem when $K\rho \geq 1$.
Theorem 1 and (16) prove this Theorem, when $K\rho < 1$.
Q.E.D.

<u>***Theorem 4:***</u> In real-time system with N servers, K ($K < N$) maintenance crews, r ($r \geq 2$) different channels operating under a maximum load regime, and exponentially distributed operating and maintenance times (with parameters μ_i ($i = \overline{1, r}$) and λ respectively) optimal assignment probabilities p_i^* ($i = \overline{1, r}$), which minimize system loss penalty function when $N \to \infty$ are determined as follows:

(a) If $\lambda K \geq \sum_{i=1}^{r} \mu_i$, then $p_j^* = \mu_j \Big/ \sum_{i=1}^{r} \mu_i$ for $j = \overline{1, r}$.

(b) If $\lambda K < \sum_{i=1}^{r} \mu_i$, then

$$p_j^* = \begin{cases} \mu_j / \lambda K, \text{ if } \sum_{i=1}^{j} \frac{\mu_i}{\lambda K} \leq 1, \\ 1 - p_1^* - \dots - p_{j-1}^*, \text{ if } \sum_{i=1}^{j-1} \frac{\mu_i}{\lambda K} \leq 1 \ \& \ \sum_{i=1}^{j} \frac{\mu_i}{\lambda K} > 1, \\ 0, \text{ if } \sum_{i=1}^{j-1} \frac{\mu_i}{\lambda K} > 1, \end{cases}$$

where the channels are numbered such that $\frac{C_1}{\mu_1} \geq \frac{C_2}{\mu_2} \geq \dots \geq \frac{C_r}{\mu_r}$.

Corresponding optimal value of loss penalty function is

(a) $TC(\rho_1^*, \dots, \rho_r^*) = 0$, if $\lambda K \geq \sum_{i=1}^{r} \mu_i$;

(b) $TC(\rho_1^*, \dots, \rho_r^*) = \sum_{i=1}^{r} C_i (1 - K\rho_i^*)$, if $\lambda K < \sum_{i=1}^{r} \mu_i$,

where $\rho_i^* = \frac{\lambda p_i^*}{\mu_i}$, $i = \overline{1, r}$.

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